

# The extensions of gravitational soliton solutions with real poles

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## Abstract

We analyse vacuum gravitational “soliton” solutions with real poles in the cosmological context. It is well known that these solutions contain singularities on certain null hypersurfaces. Using a Kasner seed solution, we demonstrate that these may contain thin sheets of null matter or may be simple coordinate singularities, and we describe a number of possible extensions through them.

## 1 Introduction

The inverse scattering (BZ) technique of Belinskii and Zakharov [1] is now well known. It is essentially a solution-generating procedure for producing exact vacuum solutions of Einstein’s equations for space-times admitting two isometries. Starting from some initial “seed” solution, the technique is based on the construction of a “dressing” matrix which is a meromorphic function of a complex spectral parameter  $\lambda$ .

For the case in which a vacuum space-time admits two hypersurface-orthogonal space-like Killing vectors, the “gravitational solitons” corresponding to particular poles of the dressing matrix generally describe perturbations of the gravitational field which propagate like finite gravitational waves on some background. Here we consider gravitational soliton solutions in such space-times which correspond to real poles in the scattering matrix. As originally pointed out by Belinskii and Zakharov [1], these solutions exist in regions which are bounded by null hypersurfaces on which singularities occur. It is the purpose of this paper to reconsider the character of these singularities and the possible extensions through them.

Carr and Verdaguer [2] have considered soliton solutions in a Kasner background and have interpreted the solutions with real poles as inhomogeneous cosmologies with shock waves in which the solitons propagate away revealing the Kasner background. However, as shown by Gleiser [3] and Curir, Francaviglia and Verdaguer [4], these solutions must contain thin sheets of null matter separating the various regions. Gleiser [3] has also described alternative matter-free extensions, while Curir, Francaviglia and Verdaguer [4] have considered a real pole of arbitrary degeneracy for the diagonal “soliton” solution of

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Carmeli and Charach [5] which was shown to correspond to a real (degenerate) pole by Feinstein and Charach [6].

Others [7]–[10] have considered soliton solutions with real poles in which the seed metric is a nondiagonal vacuum Bianchi II space-time. In this case, it is similarly possible to remove the coordinate singularity on the null hypersurface, although there may again be an impulsive gravitational wave and a  $\delta$ -function in the Ricci tensor on the shock front.

In this paper, we reconsider the physical interpretation of some soliton solutions with real poles. As summarised above, these can represent gravitational shock waves in some cosmological background. However, the extension to the background is not unique even if the possibility of thin sheets of null matter is excluded. In particular, we consider the one-soliton solution with a vacuum Bianchi I seed. By extending this through the shock front in various ways, we construct a number of different global solutions.

## 2 Real pole solitons with a Kasner seed

According to the BZ technique [1], we consider a vacuum space-time with two hypersurface-orthogonal Killing vectors. In the case in which the isometries are spacelike, the metric can be written in the form

$$ds^2 = 2e^{-M} du dv - g_{ij} dx^i dx^j \quad (1)$$

where the 2-metric  $\mathbf{g}$  ( $= g_{ij}$ ) is a function of the two null coordinates  $u$  and  $v$ , and has the determinant  $|\mathbf{g}| = \alpha^2$ . Einstein's vacuum field equations require that  $\alpha$  satisfies the 2-dimensional wave equation and so can be written in the form  $\alpha = f(u) + g(v)$ , where  $f(u)$  and  $g(v)$  are arbitrary functions. It is also convenient to introduce another function  $\beta(u, v)$  which is harmonically conjugate to  $\alpha$  and given by  $\beta = f(u) - g(v)$ . In this and following sections, we will assume that the two null coordinates  $u$  and  $v$  are future-pointing.

Here, we start with an initial Kasner seed solution (denoted by a suffix zero) which can be written in the form (1) with

$$\mathbf{g}_0 = \alpha \begin{pmatrix} \alpha^p & 0 \\ 0 & \alpha^{-p} \end{pmatrix}, \quad e^{-M_0} = \frac{\alpha_u \alpha_v}{\alpha^{(1-p^2)/2}} \quad (2)$$

where  $p$  is an arbitrary parameter. This reduces to a form of the Minkowski metric when  $p = \pm 1$ . According to the BZ technique, we work with the matrix

$$\Psi_0(u, v, \lambda) = \begin{pmatrix} (\alpha^2 + 2\beta\lambda + \lambda^2)^{(1+p)/2} & 0 \\ 0 & (\alpha^2 + 2\beta\lambda + \lambda^2)^{(1-p)/2} \end{pmatrix}$$

which satisfies the appropriate equation and the condition  $\Psi_0(u, v, 0) = \mathbf{g}_0$ .

We also restrict attention here to the case in which there is a single real pole given by  $\lambda = \mu$ , where

$$\mu = \omega - \beta + \sqrt{(\omega - \beta)^2 - \alpha^2}$$

and  $\omega$  is an arbitrary real constant. (For a single real pole, we note that the alternative expression  $\mu = \omega - \beta - \sqrt{(\omega - \beta)^2 - \alpha^2}$  simply corresponds to a rotation of coordinates  $x \rightarrow y$ ,  $y \rightarrow -x$ .) Clearly the single soliton solution is only admissible in regions of space-time for which  $(\omega - \beta)^2 \geq \alpha^2$ . (If there are more than one such regions, these will normally be disjoint.) These regions will be bounded by an initial cosmological curvature singularity which occurs when  $\alpha = 0$ . We will therefore take  $\alpha > 0$ . They will also be

bounded by null hypersurfaces on which  $(\omega - \beta)^2 = \alpha^2$ . According to common usage, we will refer to these as “shock fronts”.

Since  $\alpha^2 + 2\beta\lambda + \lambda^2 = 2\omega\lambda$ , we obtain that on the pole trajectory  $\lambda = \mu$

$$\Psi_0^{-1}(u, v, \mu) = \begin{pmatrix} (2\omega\mu)^{-(1+p)/2} & 0 \\ 0 & (2\omega\mu)^{-(1-p)/2} \end{pmatrix}.$$

The procedure is now well known and, for the nondiagonal case in the region  $\omega - \beta \geq \alpha$ , the new solution can be expressed as

$$\mathbf{g} = \left( \frac{\mu}{\alpha} \mathbf{I} - \frac{(\mu^2 - \alpha^2)}{\alpha\mu} \mathbf{P} \right) \mathbf{g}_0$$

where, after introducing a new function  $s(\alpha, \beta)$  such that  $e^s = \mu/\alpha$  and the constant  $c = p \log(2\omega)$ , the matrix  $\mathbf{P}$  is given by

$$\mathbf{P} = \frac{1}{2 \cosh(ps + c)} \begin{pmatrix} e^{-ps-c} & \alpha^p \\ \alpha^{-p} & e^{ps+c} \end{pmatrix}.$$

It may be noted that a change in the values of the free soliton parameters corresponds to a rotation of coordinates. For the single soliton solution, this is not physically significant. However, when attached to other regions, changes in these parameters can represent soliton waves with different polarization. With the above expressions, the new solution is given by

$$\mathbf{g} = \frac{\alpha}{\cosh(ps + c)} \begin{pmatrix} \alpha^p \cosh[(1+p)s + c] & -\sinh s \\ -\sinh s & \alpha^{-p} \cosh[(1-p)s - c] \end{pmatrix}. \quad (3)$$

### 3 Extending the solution to a Kasner background

As noted, the above solution using (3) is only defined in the region of space-time in which  $\omega - \beta \geq \alpha > 0$ . This region is bounded by the null hypersurface  $\beta + \alpha = \omega$  on which  $\mu = \alpha$ . It can be seen that, on this boundary,  $s = 0$  and the 2-metric is the same as that of the seed. Thus, it would seem to be possible to join this metric continuously with the seed Kasner solution in the region  $\alpha \geq \omega - \beta$ , thus forming a gravitational shock front as described by Carr and Verdager [2]. It is also possible to include a one-soliton solution in the region in which  $\beta - \omega \geq \alpha > 0$ . The standard global interpretation of this solution is then as two disjointed one-soliton regions moving apart leaving an exact Kasner background. This forms a composite space-time as illustrated in figure 1 in which there is an initial curvature singularity when  $\alpha = 0$ .

However, although the 2-metric is continuous in this construction, it may have discontinuous first derivatives which could introduce an impulsive gravitational wave component on the shock front.

More seriously, we note that, using (3), the new expression for  $M$  in the region  $\omega - \beta \geq \alpha > 0$  is given by

$$e^{-M} = \frac{C \sqrt{\alpha} \cosh(ps + c)}{\sqrt{\omega - \alpha - \beta} \sqrt{\omega + \alpha - \beta}} e^{-M_0}$$

where  $C$  is an arbitrary constant. This clearly introduces a singularity on the shock front  $\beta + \alpha = \omega$ . However this can be seen to be a coordinate singularity which can be removed by a particular choice of the functions  $f(u)$ . For example, using (2), the singularity can be removed by the choice  $f = \frac{1}{2}(\omega - u^2)$ .

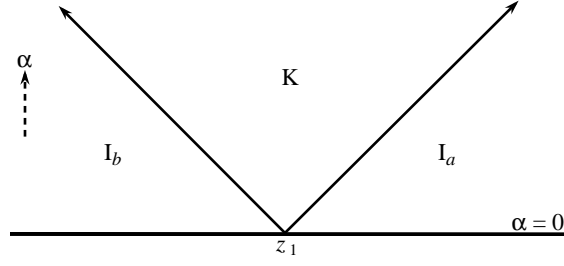


Figure 1: This represents a one-soliton solution, using  $\alpha$  and  $\beta$  as coordinates, in which the solitons propagate away to reveal a Kasner background. The space-time is a combination of two one-soliton regions (denoted by  $I_a$  and  $I_b$ ) each with a real pole  $\omega_1 = -z_1$ , together with a Kasner region (denoted by  $K$ ).

It may be recalled that the BZ method is based on the assumption that  $\alpha = f + g$  is the same for the soliton solution as for the seed metric that was used to generate it. Thus, with the choice  $f = \frac{1}{2}(\omega - u^2)$ , the seed that is used in this case is a particular form of the Kasner metric which contains a coordinate singularity on the null hypersurface  $u = 0$ . i.e. in order to generate a non-singular one-soliton solution directly, it is necessary to start with a seed solution in a form containing a coordinate singularity.

In the above construction, the space-time has been extended through the null hypersurface  $u = 0$ , on which  $\mu = \alpha$ , to a “background” region which is part of a Kasner space-time with the same parameter  $p$  as the seed solution. However, to avoid a coordinate singularity in this region, it is necessary to put  $\alpha = \frac{1}{2}(u + v)$ , which is equivalent to the choice  $f = \frac{1}{2}(u + \omega)$  and  $g = \frac{1}{2}(v - \omega)$ . In order for this to be continuous with a nonsingular soliton region, it is necessary for the soliton region to have  $f = \frac{1}{2}(\omega - u^2)$  and  $g = \frac{1}{2}(v - \omega)$  (with the region  $I_b$  the same but with  $u$  and  $v$  interchanged). From this, it follows that  $\alpha$  in the soliton regions (and the associated seed) is different to that in any extended “background” region, and that the background only has the same form as the seed solution after a coordinate transformation.

In the soliton region  $I_a$  we now have  $\alpha = \frac{1}{2}(v - u^2)$ . This has been made continuous with a Kasner background in which  $\alpha = \frac{1}{2}(u + v)$  across  $u = 0$ . However, it may be recalled that discontinuities in the derivatives of  $\alpha$  across a hypersurface induce nonzero components in the Ricci tensor, and hence in the energy-momentum tensor. In this case, these are given by

$$T_{uu} = -\frac{e^M}{8\pi\alpha} [\alpha_u] \delta(u) \quad \text{or} \quad T_{vv} = -\frac{e^M}{8\pi\alpha} [\alpha_v] \delta(v).$$

It can thus be seen that the discontinuity in the derivative of  $\alpha$  across  $u = 0$  in the above expressions gives rise to an impulsive component in the energy-momentum tensor corresponding to a thin sheet of null matter located on this hypersurface. This has been described explicitly elsewhere [3], [4]. Moreover, with this time orientation, the matter has *negative* energy density [10].

We now have a one-soliton solution (3) with a single real pole  $\omega_1 = -z_1$  in the region represented as  $I_a$  in figure 1. This has been extended to a Kasner region (represented by  $K$  in figure 1) across the shock front  $u = -z_1$ , and another one-soliton region (represented by  $I_b$ ) in which  $\omega - \beta < -\alpha$  has been added together with a second shock front  $v = z_1$ . The metric function  $\alpha = f(u) + g(v)$  is determined by the following expressions in each region:

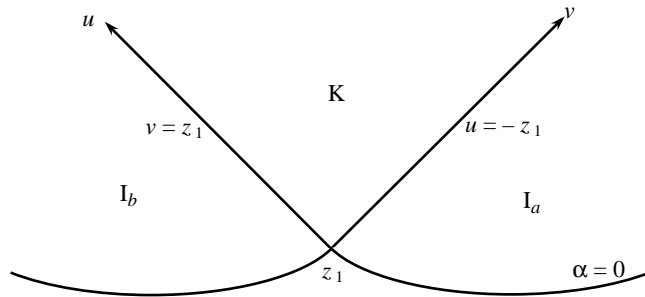


Figure 2: This represents the one-soliton solution in which the solitons propagate away to reveal a Kasner background in the  $u, v$  coordinates. An initial “cosmological” curvature singularity occurs when  $\alpha = 0$ .

K:	$u > -z_1, \quad v > z_1$	$f = \frac{1}{2}u$	$g = \frac{1}{2}v$
I <sub>a</sub> :	$u \leq -z_1, \quad v > z_1$	$f = \frac{1}{2}[-z_1 - (u + z_1)^2]$	$g = \frac{1}{2}v$
I <sub>b</sub> :	$u > -z_1, \quad v \leq z_1$	$f = \frac{1}{2}u$	$g = \frac{1}{2}[z_1 - (z_1 - v)^2]$

Clearly there are discontinuities in the derivatives of  $\alpha$  across the junctions  $u = -z_1$  and  $v = z_1$ . Thus, as described above, this configuration must contain thin sheets of null matter on these hypersurfaces.

It may also be noted that the curvature singularity at  $\alpha = 0$  is located, in these coordinates, on  $v - z_1 - (u + z_1)^2 = 0$  for  $v > z_1$  and on  $u + z_1 - (z_1 - v)^2 = 0$  for  $v \leq z_1$ . Thus, although this singularity is spacelike, it approaches a null limit at the junction point  $v = -u = z_1$ . This is illustrated in figure 2.

It is appropriate to work with a null tetrad such that  $\ell_i = e^{-M/2}u_{,i}$  and  $n_i = e^{-M/2}v_{,i}$ . Since  $M$  is continuous across the shock fronts in these coordinates, these null vectors are well behaved throughout the space-time. Using these, it can then be shown that (provided  $p \neq 0, \pm 1$ ) the Weyl tensor has non-zero components  $\Psi_0, \Psi_2$  and  $\Psi_4$  in both the soliton and Kasner regions. Moreover, these components are bounded near the shock fronts, although they are not necessarily continuous across the shock which may also have step changes in the polarization into the soliton regions (in the nondiagonal case). In addition, the shock fronts themselves may also contain impulsive components.

## 4 Soliton solutions with distinct real poles

In this section we re-consider the  $n$ -soliton solutions with distinct real poles which have been described briefly elsewhere [2] and interpreted as solitons moving apart leaving an exact Kasner background. To be specific, we concentrate on the case of a soliton solution with two distinct real poles  $\omega_1 = -z_1$  and  $\omega_2 = -z_2$ , where  $z_2 > z_1$ . This can again be interpreted as a composite space-time having one- and two-soliton regions and a Kasner background as illustrated in figure 3 using  $\alpha, \beta$  coordinates. In this case, the shock fronts can be taken to be  $u = -z_1, u = -z_2, v = z_1$  and  $v = z_2$ .

As described in the previous section, singularities can occur in the metric coefficient  $e^{-M}$ . For the two-soliton solution this is given by

$$e^{-M} = \frac{C \alpha^{((2-p)^2-1)/2} (\mu_1 \mu_2)^{2+p} \alpha_u \alpha_v}{(\alpha^2 - \mu_1^2)(\alpha^2 - \mu_1 \mu_2)^2 (\alpha^2 - \mu_2^2)}.$$

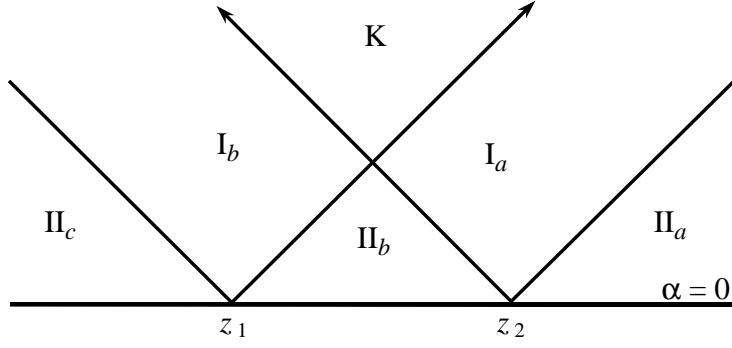


Figure 3: This represents soliton solutions with two real poles with  $\alpha$  and  $\beta$  as coordinates. The solitons propagate away to reveal a Kasner background. The space-time is a combination of one- and two-soliton regions (denoted by  $I_a$ ,  $I_b$ ,  $II_a$ ,  $II_b$ ,  $II_c$ ) with real poles  $\omega_1 = -z_1$  and  $\omega_2 = -z_2$ , together with a Kasner region (denoted by  $K$ ).

An explicit transformation which removes each singularity individually has been given by Díaz and Gleiser [11] (for the general case of  $n$  distinct real poles). However, this can only be applied to one singularity at a time as the resulting metric becomes discontinuous on the next singular wavefront. Below, we introduce a different gauge in which the metric is continuous throughout the space-time.

The two-soliton solution is appropriate in the regions  $II_a$ ,  $II_b$  and  $II_c$  indicated in figure 2. It is then possible to extend the solution into regions described by the one-soliton solution (3) and then further to the Kasner solution (2) as illustrated. We can adopt two future-pointing null coordinates  $u$  and  $v$  that are defined globally. It is then possible to introduce a gauge such that  $f(u)$  and  $g(v)$ , and hence  $\alpha(u, v)$  and  $\beta(u, v)$  are continuous everywhere. One such gauge is as follows:

$$\begin{aligned}
K: \quad & u > -z_1, \quad v > z_2, \quad \begin{cases} f = \frac{1}{2}u, \\ g = \frac{1}{2}v \end{cases} \\
I_a: \quad & -z_1 \geq u \geq -z_2, \quad v > z_2, \quad \begin{cases} f = \frac{1}{2}[-z_1 - (u + z_1)^2 - k(u + z_1)^3], \\ g = \frac{1}{2}v \end{cases} \\
I_b: \quad & u > -z_1, \quad z_2 \geq v > z_1, \quad \begin{cases} f = \frac{1}{2}u, \\ g = \frac{1}{2}[z_2 - (z_2 - v)^2 + k(z_2 - v)^3] \end{cases} \\
II_a: \quad & -z_2 \geq u, \quad v > z_2, \quad \begin{cases} f = \frac{1}{2}[-z_2 - (u + z_2)^2], \\ g = \frac{1}{2}v \end{cases} \\
II_b: \quad & -z_1 \geq u \geq -z_2, \quad z_2 \geq v > z_1, \quad \begin{cases} f = \frac{1}{2}[-z_1 - (u + z_1)^2 - k(u + z_1)^3], \\ g = \frac{1}{2}[z_2 - (z_2 - v)^2 + k(z_2 - v)^3] \end{cases} \\
II_c: \quad & u > -z_1, \quad z_1 \geq v, \quad \begin{cases} f = \frac{1}{2}u, \\ g = \frac{1}{2}[z_1 - (z_1 - v)^2] \end{cases}
\end{aligned}$$

where  $k = (z_2 - z_1)^{-1} - (z_2 - z_1)^{-2}$ .

With these expressions, it can be seen that the metric is  $C^0$  everywhere and that all coordinate singularities have been removed. However, it can also be seen that there are discontinuities in the derivatives of  $\alpha$  on the null hypersurfaces  $u = -z_1$ ,  $u = -z_2$ ,  $v = z_1$

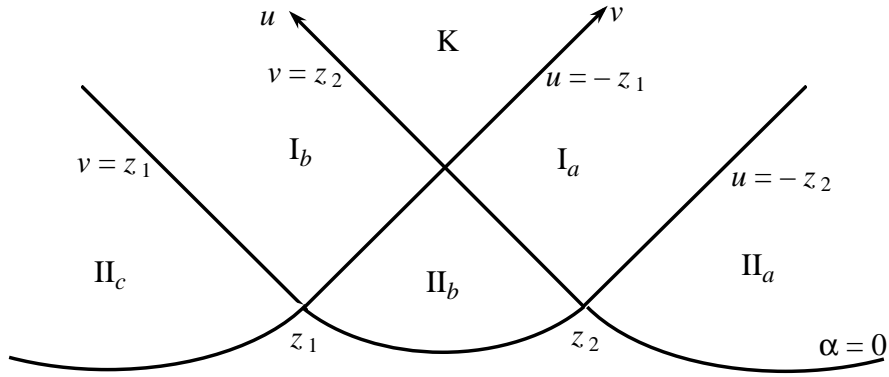


Figure 4: This represents soliton solutions with two real poles in the  $u, v$  coordinates. An initial “cosmological” curvature singularity occurs when  $\alpha = 0$ .

and  $v = z_2$  which therefore must contain thin sheets of null matter. The structure in  $u, v$  coordinates is illustrated in figure 4.

It may be noted that the space-times in the soliton regions may be diagonal or non-diagonal. However, some soliton parameters must be continued across the shock fronts. To be explicit, the one-soliton regions must contain solitons whose parameters are continued from the two-soliton regions  $II_a$  or  $II_c$ . The region  $II_b$ , however, must contain a continuation of the two solitons which extend back from regions  $I_a$  and  $I_b$  and therefore its parameters are predetermined from those of  $II_a$  and  $II_c$ .

Clearly, this approach can be generalised to include an arbitrary number of distinct real poles. For solutions with more than one real pole, these solutions necessarily contain thin sheets of null matter with negative energy density on the boundaries between the various regions.

## 5 Possible extensions without sheets of null matter

We now consider the possible extensions of the one-soliton solution which do not involve thin sheets of null matter. In this case, no generality is lost in making a coordinate shift to put  $\omega = 0$  so that the shock front of the soliton region is then given by  $u = 0$ . We also continue to use two future-pointing null coordinates  $u$  and  $v$ .

We may initially consider whether or not it is possible to construct an exact one-soliton solution which has the same global structure as that indicated in Figure 1, but without the presence of thin sheets of null matter. For this, the regions  $I_a$  and  $I_b$  would be essentially the same, but the extension would not be to a Kasner background. If, in the one-soliton region  $I_a$  we put  $\alpha = \frac{1}{2}(v^2 - u^2)$ , it may be possible in the extended region to choose a gauge such that  $\alpha = \frac{1}{2}(u^2 + v^2)$ . However, it is shown in the appendix that such an extension is not possible. We therefore look for alternative extensions to the region  $I_a$ . Since the shock front is null, any extension will be non-unique. In fact, a number of possibilities are readily available.

It may first be noted that the one-soliton region is algebraically general. There will therefore be a gravitational wave component propagating towards and through the shock front. In the soliton region, there will also be a gravitational wave component propagating in the opposite direction parallel to the shock front. However, this component need not occur in the extended region and, in this case, the extension will be to a plane wave region. Other possibilities include that in which the extension is to another distinct soliton region, or to that containing an arbitrary gravitational wave component propagating parallel to

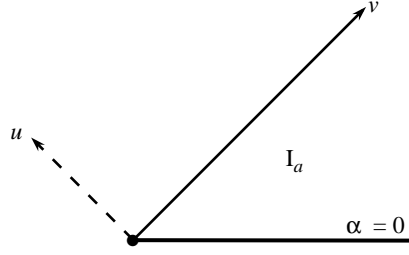


Figure 5: A one-soliton region can be extended to a plane wave region. Gravitational wave components originating in the singularity propagate into the plane wave region but, in that region, no wave components propagate along the lines  $u = \text{const.}$  A non-scalar curvature singularity occurs on  $v = 0$ .

the shock front (although only the linear case will be constructed below).

## 5.1 A plane wave extension

The simplest extension across the shock front  $u = 0$  is to a plane wave region in which  $f = 0$  and  $\mathbf{g}$  and  $M$  are functions of  $v$  only. The general structure is illustrated in figure 5.

For the moment, let us continue to adopt a gauge such that  $g(v) = \frac{1}{2}v$ . In the extension to a plane wave region, we may now set  $f(u) = 0$  and  $\alpha = -\beta = \frac{1}{2}v$  and the metric is given by

$$e^{-M} = c_1 v^{(p^2-1)/2}, \quad \mathbf{g} = \frac{v}{2} \begin{pmatrix} (\frac{v}{2})^p & 0 \\ 0 & (\frac{v}{2})^{-p} \end{pmatrix}.$$

It is then appropriate to make the coordinate transformation

$$v = \tilde{v}^{2/(p^2+1)}$$

together with a rescaling to remove an unwanted constant so that the line element becomes

$$ds^2 = 2du d\tilde{v} - \tilde{v}^{2/(p^2+1)} \left( \tilde{v}^{2p/(p^2+1)} dx^2 + \tilde{v}^{-2p/(p^2+1)} dy^2 \right).$$

We can then make the further coordinate transformation

$$\begin{aligned} X &= \tilde{v}^{\frac{1+p}{1+p^2}} x \\ Y &= \tilde{v}^{\frac{1-p}{1+p^2}} y \\ r &= u + \frac{1}{2} \left( \frac{1+p}{1+p^2} \right) \tilde{v}^{\frac{1+2p-p^2}{1+p^2}} x^2 + \frac{1}{2} \left( \frac{1-p}{1+p^2} \right) \tilde{v}^{\frac{1-2p-p^2}{1+p^2}} y^2 \end{aligned}$$

to put the line element in the form

$$ds^2 = 2d\tilde{v} dr - \frac{p(1-p^2)}{(1+p^2)^2} \tilde{v}^{-2} (X^2 - Y^2) d\tilde{v}^2 - dX^2 - dY^2$$

which is the familiar form of a plane wave with amplitude profile  $h(\tilde{v}) = \frac{p(1-p^2)}{(1+p^2)^2} \tilde{v}^{-2}$ . Thus, except for the cases in which  $p = 0, \pm 1$  in which the extension is to a flat region, the plane gravitational wave amplitude in the extended region is clearly unbounded when  $\tilde{v} = 0$  (or  $v = 0$ ). This null hypersurface may then reasonably be considered to form a boundary of the space-time extended from the one-soliton solution as described in figure 5.

It may be observed that this situation is qualitatively identical to the time reverse of a colliding plane wave space-time in which a future curvature singularity is formed following the interaction of initially plane waves. As in that case, the non-scalar curvature singularity that occurs in the plane wave region on  $v = 0$  can be interpreted as a “fold singularity” as described by Matzner and Tipler [12] (see also [13]).



## 5.2 A soliton extension

An alternative extension can be achieved by matching possibly different one-soliton solutions on either side of the shock front. On either side we can adopt the same gauge with  $f = -\frac{1}{2}u^2$  and  $g = \frac{1}{2}v^2$ , so that

$$\alpha = \frac{1}{2}(v^2 - u^2) \quad \text{and} \quad \beta = -\frac{1}{2}(u^2 + v^2). \quad (4)$$

This clearly satisfies the required inequality for the one-soliton solution. Taking  $u$  to be time-oriented, we now have two regions. An initial region with  $u \leq 0$  and a second region with  $u \geq 0$ .

It is also appropriate here to introduce alternative coordinates  $t$  and  $\rho$  where

$$t = \frac{1}{\sqrt{2}}(u + v) \quad \text{and} \quad \rho = \frac{1}{\sqrt{2}}(v - u),$$

so that  $\alpha = t\rho$  and  $\mu = t^2$ .

The curvature singularity at  $\alpha = 0$  now occurs both when  $t = 0$  and when  $\rho = 0$  so that the space-time is defined only in regions for which  $v > |u|$ . However, this includes two regions in which the conditions for a one-soliton solution are satisfied. These occur on either side of the shock front as illustrated in figure 6. Of course, there is no reason for solitons in these two regions to have the same parameters, and it is possible to construct a compound space-time composed of two different one-soliton solutions joined across the shock front. For example, we can choose the soliton in one region to be diagonal, and that in the other non-diagonal. In addition, since  $f$  and  $g$  are taken to have the same form in both regions, there is no discontinuity in the derivatives of  $\alpha$ , and so there will be no sheets of null matter across the shock front. This matter-free extension of the one-soliton solution was initially pointed out by Gleiser [3] at least for the case when  $p = -1$ .

As a particular example, let us consider the case of the one-soliton solutions that are generated from the plane symmetric (type D) Kasner solution for which  $p = 0$ . The non-diagonal case is given by

$$\mathbf{g} = t\rho \begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix} \quad \text{where} \quad e^s = t/\rho. \quad (5)$$

This solution can be adopted in the region  $t \geq \rho > 0$ . Choosing  $C = -\omega$ , the remaining part of the metric is given by

$$2e^{-M} du dv = dt^2 - d\rho^2. \quad (6)$$

Let us now attach this to a background region  $u < 0$  and  $v > 0$  (i.e.  $\rho > t \geq 0$ ) which is composed of the diagonal one-soliton solution with a  $p = 0$  Kasner seed, given by

$$\mathbf{g} = t\rho \begin{pmatrix} t/\rho & 0 \\ 0 & \rho/t \end{pmatrix}, \quad e^{-M} = 1.$$

This can be expressed in the form

$$ds^2 = dt^2 - d\rho^2 - t^2 dz^2 - \rho^2 d\phi^2 \quad (7)$$

which after the coordinate transformation

$$T = t \cosh z, \quad X = \rho \cos \phi, \quad Y = \rho \sin \phi, \quad Z = t \sinh z$$

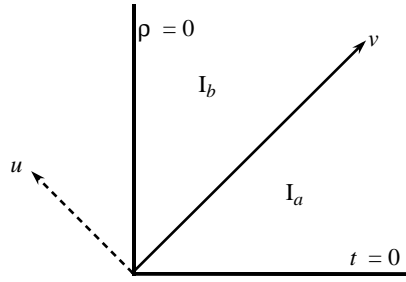


Figure 6: This space-time is composed of two (possibly different) one-soliton solutions in the two regions denoted by  $I_a$ ,  $I_b$ . It is defined only for  $v^2 - u^2 > 0$ . Generally, there are curvature singularities on the hypersurfaces  $t = 0$  and  $\rho = 0$ .

is clearly that part  $T \geq |Z|$  of the Minkowski space-time

$$ds^2 = dT^2 - dX^2 - dY^2 - dZ^2$$

in cartesian coordinates. In this case, it can be seen that the junction between the two regions, which is the null hypersurface  $u = 0$  (or  $t = \rho$ ), is an expanding sphere given by  $T^2 - X^2 - Y^2 - Z^2 = 0$ . The solution therefore describes a gravitational wave with an exact spherical wavefront propagating into a Minkowski background.

It may also be observed that the metric (5–6) may be written in the form

$$ds^2 = dt^2 - d\rho^2 - \frac{1}{2}t^2(dz - d\phi)^2 - \frac{1}{2}\rho^2(dz + d\phi)^2$$

which is clearly simply a rotation of the flat metric (7). Thus, the space-time inside the spherical wavefront is just another version of Minkowski space. The single soliton solution obtained from a plane symmetric Kasner seed is a flat space-time. The only possible nonzero components of the Weyl tensor arise from the discontinuities in the derivatives of the metric across the wavefront. The solution described above therefore represents an impulsive spherical wave propagating in a Minkowski background. This is in fact a special case of the impulsive spherical gravitational wave that was constructed by Penrose [14] using a “cut and paste” method.

It may also be observed in this case that  $t = 0$  simply corresponds to a coordinate singularity in the Minkowski background. It is thus possible to add the Minkowski region in which  $0 \leq T < |Z|$ , and then to add the time-reverse of the solution for  $T < 0$  and  $t < 0$ . The global solution then describes a contracting impulsive gravitational wave in a Minkowski background which collapses to a point. After the collapse, the wave then re-expands as an exact spherical impulsive wave. The only singularity occurs at the event at which the spherical wave has zero radius.

It may also be pointed out that Gleiser, Garate and Nicassio [15] have obtained a similar singularity-free solution in a one-soliton solution generated from a Bianchi VI<sub>0</sub> seed. In that work they describe the soliton perturbation as “erasing” the “cosmological” singularity that occurs in the seed.

For the case of a more general Kasner seed, solutions can be constructed as outlined above. For  $p \neq 0$ , they will have curvature singularities both when  $t = 0$  and when  $\rho = 0$ , so that the complete space-time is the region  $t > 0$ ,  $\rho > 0$ . However, these cases do not strictly have an axis at  $\rho = 0$  and the space is not asymptotically flat.

### 5.3 A non-soliton extension

In the plane wave extension of §5.1, the gravitational waves propagating in the direction  $v = \text{const.}$  simply continue into the plane wave region in which no waves propagate

in the opposite direction. However, it is natural to consider the possible existence of gravitational waves propagating in the extended region parallel to the shock front. Such components occur in the soliton extension of §5.2 in which both regions are algebraically general, although only specific wave profiles are permitted. We now consider an alternative extension which includes arbitrary gravitational wave components in the extended region — at least for the diagonal case in which such waves have constant aligned polarization. Such an extension can be constructed as follows.

We start by choosing coordinates in the one-soliton region with  $f = -\frac{1}{2}u^2$  and  $g = \frac{1}{2}v^2$  so that  $\alpha$  and  $\beta$  are given by (4). We then adopt the same expression for  $\alpha(u, v)$  in the extended region so that there are no Ricci tensor components across the shock front. The metric in the extended region can then be taken in the form

$$ds^2 = 2e^{-M} du dv - \alpha(e^V dx^2 + e^{-V} dy^2),$$

where

$$V(u, v) = p \log \alpha + \tilde{V}(u, v)$$

and  $\tilde{V}$  is an arbitrary function satisfying  $\tilde{V}(0, v) = 0$ , so that the metric remains continuous across the shock front.

An expression for  $\tilde{V}$  satisfying the above property is given by the Rosen pulse solution [16] which, in this context, takes the form

$$\tilde{V} = \int_0^f \frac{F(\sigma) d\sigma}{\sqrt{\sigma - f} \sqrt{\sigma + g}} = - \int_{-u^2}^0 \frac{F(\sigma) d\sigma}{\sqrt{\sigma + u^2} \sqrt{\sigma + v^2}},$$

where  $F(\sigma)$  is an arbitrary function. In the present situation we require  $F(0) = 0$  and the continuity of  $F$  at  $\sigma = 0$  must be determined very carefully. However, since a complete solution cannot be determined for the solution in this form, its application here is very limited.

On the other hand, an explicit representation for  $\tilde{V}$  for which a complete solution can be obtained, and which satisfies the required properties above, has been given in [17]–[19]. These solutions are expressed as a sum over explicit components, each of which have the self-similar form

$$\tilde{V}_k(f, g) = (f + g)^k H_k\left(\frac{g-f}{f+g}\right)$$

where  $k$  is an arbitrary real parameter. Putting  $\zeta = -\beta/\alpha$ , the functions  $H_k(\zeta)$  satisfy the linear equation

$$(\zeta^2 - 1)H_k'' + (1 - 2k)\zeta H_k' + k^2 H_k = 0,$$

together with the initial condition  $H_k(1) = 0$ , so that they satisfy the recursion relations

$$H_k(\zeta) = \int_1^\zeta H_{k-1}(\zeta') d\zeta' \quad \text{or} \quad H_k'(\zeta) = H_{k-1}(\zeta).$$

These solutions can be expressed in terms of standard hypergeometric functions in the form

$$(f + g)^k H_k\left(\frac{g-f}{f+g}\right) = c_k \frac{f^{1/2+k}}{\sqrt{f+g}} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2} + k; \frac{f}{f+g}\right)$$

where, for integer  $k$ ,  $c_k = (-1)^k 2^k \Gamma(\frac{3}{2})/\Gamma(k + \frac{3}{2})$ . The papers [17]–[19], however, are concerned with gravitational waves with distinct wavefronts propagating into certain simple backgrounds. For such situations, it is only necessary to consider cases in which  $k \geq \frac{1}{2}$ . However, in the present situation, it is necessary to choose the minimum value of  $k$  in order to remove the singularity that would otherwise arise in the metric coefficient  $e^{-M}$ , and it can be seen that this must involve the component  $k = 0$ .

In determining an explicit solution for the extension, we may use the fact that

$$H_0(\zeta) = \cosh^{-1} \zeta \quad \text{and} \quad H_{-1}(\zeta) = \frac{1}{\sqrt{\zeta^2 - 1}}.$$

It is convenient initially to consider the solution  $\tilde{V} = a_0 H_0(\zeta)$ , where  $a_0$  is a constant, so that

$$\begin{aligned} V &= p \log \alpha + a_0 H_0(\zeta) \\ &= p \log(f + g) + a_0 H_0\left(\frac{g-f}{f+g}\right). \end{aligned}$$

The remaining vacuum field equations as given in [17] can then be integrated, yielding

$$\begin{aligned} e^{-M} &= \frac{C|f'g'|(\zeta + \sqrt{\zeta^2 - 1})^{a_0 p} \alpha^{(p^2 - a_0^2 - 1)/2}}{(\zeta^2 - 1)^{a_0^2/2}} \\ &= \frac{C|f'g'|(\sqrt{g} + \sqrt{-f})^{2a_0 p} (f + g)^{[(a_0 - p)^2 - 1]/2}}{(-4fg)^{a_0^2/2}}, \end{aligned}$$

where  $C$  is an arbitrary constant. With the above expressions for  $f$  and  $g$ , it can immediately be seen that a coordinate singularity is avoided only if  $a_0 = \pm 1$ . Taking  $a_0 = 1$  and a particular value for  $C$ , we obtain that

$$e^{-M} = \left(\frac{v + u}{v - u}\right)^p (v^2 - u^2)^{p^2/2},$$

indicating that the metric is now continuous as required across the shock front. However, it may be noticed that this case in which  $\tilde{V} = H_0(\zeta)$  is just the diagonal case of the one-soliton solution, and therefore belongs to the class of extensions discussed in the previous subsection.

A more general extension in which the metric is diagonal in the extended region can now be constructed using

$$V = p \log \alpha + \sum_{n=0}^{\infty} a_n \alpha^n H_n(\zeta),$$

where  $a_0 = 1$  to avoid the coordinate singularity on the shock front, and the remaining coefficients  $a_n$  are arbitrary. In this case, it can be shown that

$$M = -\frac{p^2}{2} \log \alpha - p \sum_{n=0}^{\infty} a_n \alpha^n H_n(\zeta) - \sum_{n=1}^{\infty} \frac{1}{2n} \alpha^n K_n(\zeta)$$

where

$$K_n(\zeta) = \sum_{k=0}^{n-1} a_k a_{n-k} \left[ k(n-k) H_k H_{n-k} - (\zeta^2 - 1) H_{k-1} H_{n-k-1} \right].$$

This solution includes additional arbitrary gravitational wave components beyond the shock front. However, we note that the curvature singularity which occurs when  $\alpha = 0$  is now located on the spacelike hypersurface  $u + v = 0$  in the soliton region and on the timelike hypersurface  $v - u = 0$  in the extended region. Since this is a curvature singularity, it must form a boundary to the extended space-time as illustrated in figure 6.

## 6 Discussion

In the above sections, we have considered the physical interpretation and possible extensions for the soliton solutions with real poles in the case when the seed solution is taken to be the Bianchi I vacuum Kasner solution. Very different solutions can be constructed using alternative seed solutions. However, the character of the shock front and the possible extensions across it is likely to have some similar properties in all cases.

At least for solitons with a Kasner seed, we have clarified the character of the singularity that occurs on the shock front, and we have demonstrated a number of possible extension across it. The occurrence of thin sheets of null matter, and possible soliton extensions, have been discussed in previous literature for some cases. We have clarified here the structure of a plane wave extension and given a new explicit vacuum extension in which the metric is diagonal. There are clear problems in generalising the non-soliton extension to the non-diagonal case as the equations are then non-linear and superposition does not apply. Thus, the full class of permissible extensions has still not been determined.

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## Appendix

Here we consider whether it is possible to construct an exact one-soliton solution with the same global structure as that indicated in Figure 1, but without the presence of thin sheets of null matter. Taking the regions  $I_a$  and  $I_b$  to be the same, the extension would not be Kasner. Taking  $\alpha = \frac{1}{2}(v^2 - u^2)$  in the one-soliton region  $I_a$ , it is appropriate to consider whether it is possible to choose a gauge such that  $\alpha = \frac{1}{2}(u^2 + v^2)$  in the extended region. This is clearly  $C^1$  across  $u = 0$ , so the Ricci tensor vanishes on this null hypersurface.

Essentially, we can now prove that solutions of the type outlined above do not exist. The proof of these statements are roughly as follows: We start with the general metric for a space-time with two spacelike hypersurface orthogonal Killing vectors in the form

$$ds^2 = 2e^{-M} du dv - (f(u) + g(v)) (\chi dy^2 + \chi^{-1} (dx - \omega dy)^2)$$

where the coefficients depend on  $u$  and  $v$  (or  $f$  and  $g$ ) only. The vacuum field equations imply that

$$e^{-M} = \frac{|f'g'|}{\sqrt{f+g}} e^{-S}, \quad (8)$$

where

$$S_f = -\frac{1}{2}(f+g) \frac{(\chi_f^2 + \omega_f^2)}{\chi^2}, \quad S_g = -\frac{1}{2}(f+g) \frac{(\chi_g^2 + \omega_g^2)}{\chi^2}. \quad (9)$$

We may now adopt a gauge such that

$$f(u) = \frac{1}{2}\epsilon u^2, \quad g(v) = \frac{1}{2}v^2.$$

There is no loss of generality in adopting this form for  $g(v)$  and the freedom  $u \rightarrow u'(u)$  has been used to obtain the simplest expression for which  $f(0) = 0$ , and  $f'(0) = 0$  to avoid any null matter on  $u = 0$ . The freedom in rescaling  $u$  can be further used to set  $\epsilon = \pm 1$ . In this case we have

$$f' = \epsilon u.$$

Thus (8) implies that, for  $M$  to be continuous across the front  $u = 0$  (on which  $f + g > 0$ ),  $S$  must contain the term  $\log u$ . i.e. near the front  $u = 0$ ,  $S$  must behave as

$$\begin{aligned} S &\sim \log u + \text{const.} + \dots \\ &\sim \frac{1}{2} \log |f| + \dots \end{aligned}$$

Thus

$$S_f \sim \frac{1}{2f} + \dots$$

It can then be seen that the first equation in (9) can only be satisfied near the wavefront if  $f < 0$ . i.e. it is necessary that

$$\epsilon = -1.$$

Thus, solutions with  $\alpha = u^2 + v^2$  do not exist near  $u = 0$ . Further, for a vacuum extension, it is always possible to choose a gauge such that  $\alpha = v^2 - u^2$ .

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